Theta-function parametrization and fusion for 3D integrable Boltzmann weights

2004 J. Phys. A: Math. Gen. 37 1159

(http://iopscience.iop.org/0305-4470/37/4/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 1159–1179 PII: S0305-4470(04)70239-5

Theta-function parametrization and fusion for 3D integrable Boltzmann weights

G von Gehlen[1], **S Pakuliak**[2,3] **and S Sergeev**[2,4,5]

[1] Physikalisches Institut der Universität Bonn, Nussallee 12, D-53115 Bonn, Germany
[2] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow region, Russia
[3] Institute of Theoretical and Experimental Physics, Moscow 117259, Russia
[4] Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
[5] Department of Theoretical Physics, Building 59, Research School of Physical Sciences and Engineering, The Australian National University, Canberra, ACT 0200, Australia

E-mail: gehlen@th.physik.uni-bonn.de, pakuliak@thsun1.jinr.ru and sergey.sergeev@anu.edu.au

Received 10 October 2003
Published 9 January 2004
Online at stacks.iop.org/JPhysA/37/1159 (DOI: 10.1088/0305-4470/37/4/005)

Abstract

We report progress in constructing Boltzmann weights for integrable three-dimensional lattice spin models. We show that a large class of vertex solutions to the modified tetrahedron equation (MTE) can be conveniently parametrized in terms of Nth roots of theta functions on the Jacobian of a compact algebraic curve. Fay's identity guarantees the Fermat relations and the classical equations of motion for the parameters determining the Boltzmann weights. Our parametrization allows us to write a simple formula for fused Boltzmann weights $\mathfrak{R}$ which describe the partition function of an arbitrary open box and which also obey the modified tetrahedron equation. Imposing periodic boundary conditions we observe that the $\mathfrak{R}$ satisfy the normal tetrahedron equation. The scheme described contains the Zamolodchikov–Baxter–Bazhanov model and the chessboard model as special cases.

PACS numbers: 05.45.−a, 05.50.+q
Mathematics Subject Classification: 82B23, 70H06

Introduction

The tetrahedron equation is the three-dimensional generalization of the Yang–Baxter equation which guarantees the existence of commuting transfer matrices. The importance of Yang–Baxter equations for modern mathematics and mathematical physics is well known. However, the nature of the tetrahedron equation is much less understood, mainly because it is a much more complicated equation.

Given the physical interest in understanding the nature of the singularities which give rise to 3D phase transitions, any effort which gets us closer to analytic results for 3D statistical systems seems worthwhile. What has been achieved recently is to construct large classes of 3D solvable models with $\mathbb{Z}_N$-spin variables and to streamline the otherwise complicated formalism. Much more work is needed to find analytic results for partition functions and order parameters. The only available result from Baxter [26] does not lend itself to generalizations.

The first solution of the tetrahedron equation was obtained in 1980 by Zamolodchikov [1, 2] and then generalized by Baxter and Bazhanov [3] (ZBB model) and others [4]. These models have $\mathbb{Z}_N$-spin variables and solve the IRC (interaction round a cube) version of the tetrahedron equation. Later the solution of the dual equation, the vertex tetrahedron equation, was also obtained [5], generalizing several vertex solutions known previously [6, 7]. Here we shall consider only vertex-type solutions, which are usually denoted by $\mathbf{R}$, the symmetry will include a $\mathbb{Z}_N$. In general these $\mathbf{R}$-matrices obey the so-called 'simple modified tetrahedron equation' which has recently been investigated in [9]. The modified tetrahedron equation (MTE) allows us to obtain the ordinary tetrahedron equation for composite weights or vertices. In the IRC formulation this has been shown in [10, 11], while the most simple vertex case was considered in [12].

In this paper we shall introduce a new convenient theta-function parametrization of general $\mathbf{R}$ operators. This parametrization will allow us to define fused weights $\mathfrak{R}$, which are partition functions of open cubes of size M^3, and which obey a MTE. In special cases the $\mathfrak{R}$ solve an ordinary tetrahedron equation.

The vertex matrix $\mathfrak{R} \in \mathrm{End}\left(\mathbb{C}^{3NM^2}\right)$ is parametrized in terms of Nth roots of theta functions on the Jacobian of a genus $g = (M-1)^2$ compact algebraic curve Γ_g. The divisors of three meromorphic functions on Γ_g play the role of the spectral parameters for $\mathfrak{R}$. An additional parameter of $\mathfrak{R}$ is an arbitrary $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$. The tetrahedron equation for $\mathfrak{R}$ holds due to M^4 simple modified tetrahedron equations. In the case when $M = 1$ and therefore $\Gamma_g = S_2$, the solution of the simple tetrahedron equation of [5] is reproduced.

This paper is organized as follows. In section 1 we recall the vertex formulation of the 3D integrable ZBB model and sketch the derivation of the matrix operator $\mathbf{R}_{ijk}$ from a current conservation principle and Z-invariance. It satisfies the MTE and can be parametrized by quadrangle line sections. In section 2 we introduce the parametrization of $\mathbf{R}_{ijk}$ in terms of theta functions. The Fermat relation and the Hirota equations are written as Fay identities. In section 3 we show that a theta-function parametrization allows a compact formulation of the fusion of many $\mathbf{R}_{ijk}$ to the Boltzmann weight $\mathfrak{R}$ of a whole open cube which satisfies a MTE. In section 4 we first consider the special case of vanishing Jacobi transforms, in which $\mathfrak{R}$ satisfies a simple TE, then we discuss the rational case and the relation to chessboard models. Finally, section 5 summarizes our conclusions.

1. The R-matrix and its parametrization

In this first section we give a short summary of basic previous results which also serves to fix the notation.

1.1. The **R***-matrix of the vertex ZBB model*

We start recalling the vertex formulation of the ZBB model [5]. We consider a three-dimensional lattice with the elementary cell defined by three non-coplanar vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$

and general vertices

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \qquad n_1, n_2, n_3 \in \mathbb{Z}. \tag{1}$$

We label the directed link along $\mathbf{e}_j$ starting from $\mathbf{n}$ by $(j, \mathbf{n})$. On these links there are spin variables $\sigma_{j,\mathbf{n}}$ which take values in $\mathbb{Z}_N$. The partition function is defined by

$$Z = \sum_{\{\sigma\}} \prod_{\mathbf{n}} \langle \sigma_{1,\mathbf{n}}, \sigma_{2,\mathbf{n}+\mathbf{e}_2}, \sigma_{3,\mathbf{n}} | \mathbf{R} | \sigma_{1,\mathbf{n}+\mathbf{e}_1}, \sigma_{2,\mathbf{n}}, \sigma_{3,\mathbf{n}+\mathbf{e}_3} \rangle \tag{2}$$

where $\mathbf{R}$ is an operator (which in the ZBB model is independent of $\mathbf{n}$) mapping the initial three spin variables to the three final ones, so that

$$R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'} \equiv \langle \sigma_1, \sigma_2, \sigma_3 | \mathbf{R} | \sigma_1', \sigma_2', \sigma_3' \rangle \tag{3}$$

is a $N^3 \times N^3$ matrix.

For the vertex ZBB model, (3) can be expressed as a kind of cross-ratio of four cyclic functions $W_p(n)$. Introduce a two component vector $p = (x, y)$ which is restricted to the Fermat curve

$$x^N + y^N = 1. \tag{4}$$

Then define the function $W_p(n)$ by

$$W_p(0) = 1 \qquad W_p(n) = \prod_{\nu=1}^{n} \frac{y}{1 - q^\nu x} \qquad \text{for} \quad n > 0 \tag{5}$$

where

$$q = \mathrm{e}^{2\pi \mathrm{i}/N} \tag{6}$$

is the primitive *N*th root of unity. Because of the Fermat curve restriction, $W_p(n)$ is cyclic in n:

$$W_p(n + N) = W_p(n).$$

Now $\mathbf{R} = \mathbf{R}(p_1, p_2, p_3, p_4)$ is defined by the following matrix function depending on four Fermat points p_1, p_2, p_3, p_4:

$$R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'} \stackrel{\text{def}}{=} \delta_{\sigma_2+\sigma_3,\sigma_2'+\sigma_3'} q^{(\sigma_1'-\sigma_1)\sigma_3'} \frac{W_{p_1}(\sigma_2 - \sigma_1) W_{p_2}(\sigma_2' - \sigma_1')}{W_{p_3}(\sigma_2' - \sigma_1) W_{p_4}(\sigma_2 - \sigma_1')} \tag{7}$$

where *x*-coordinates of four Fermat curve points in (7) are identically related by

$$x_1 x_2 = q x_3 x_4. \tag{8}$$

So the matrix elements $R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'}$ depend on three complex numbers. These correspond to Zamolodchikov's spherical angles in the IRC formulation of the ZBB model [5]. The structure of the indices of the matrix (7) allows one to consider $\mathbf{R}$ as the operator acting in the tensor product of three vector spaces

$$\mathcal{V} = \mathbb{C}^N \qquad \mathbf{R} \in \mathrm{End}\,(\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}). \tag{9}$$

It is conventional to enumerate naturally the components of the tensor product of several vector spaces, so that (7) are the matrix elements of $\mathbf{R} = \mathbf{R}_{123}$. Of course, $\mathbf{R}_{123}$ acts trivially on all other vector spaces if one considers $\mathcal{V}^{\otimes \Delta}$ for some arbitrary Δ.

Equation (7) is known as the $\mathbf{R}$-matrix of the Zamolodchikov–Bazhanov–Baxter model, see [5]. The proof that (7) satisfies the tetrahedron equation

$$\sum_{j_1,\ldots,j_6} R_{i_1 i_2 i_3}^{j_1 j_2 j_3}(p^{(1)}) R_{j_1 i_4 i_5}^{k_1 j_4 j_5}(p^{(2)}) R_{j_2 j_4 i_6}^{k_2 k_4 j_6}(p^{(3)}) R_{j_3 j_5 j_6}^{k_3 k_5 k_6}(p^{(4)})$$
$$= \sum_{j_1,\ldots,j_6} R_{i_3 i_5 i_6}^{j_3 j_5 j_6}(p^{(4)}) R_{i_2 i_4 j_6}^{j_2 j_4 k_6}(p^{(3)}) R_{i_1 j_4 j_5}^{j_1 k_4 k_5}(p^{(2)}) R_{j_1 j_2 j_3}^{k_1 k_2 k_3}(p^{(1)}) \tag{10}$$

is rather tedious [5]. In (10) the arguments $p^{(j)}$ $(j = 1, \ldots, 4)$ stand for four Fermat curve

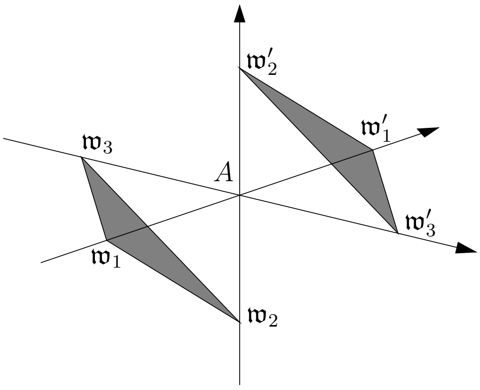

Figure 1. The six links of the basic lattice intersecting in the vertex A, intersected by auxiliary planes (shaded) in two different positions: the first passing through $\mathfrak{w}_1$, $\mathfrak{w}_2$, $\mathfrak{w}_3$ and the second through $\mathfrak{w}'_1$, $\mathfrak{w}'_2$, $\mathfrak{w}'_3$. The second position is obtained from the first by moving the auxiliary plane parallel through the vertex A. The Weyl variables, elements of $\mathfrak{w}_i$, $\mathfrak{w}'_i$, live on the links of the basic lattice. $\mathcal{R}$ maps the left auxiliary triangle onto the upper right one.

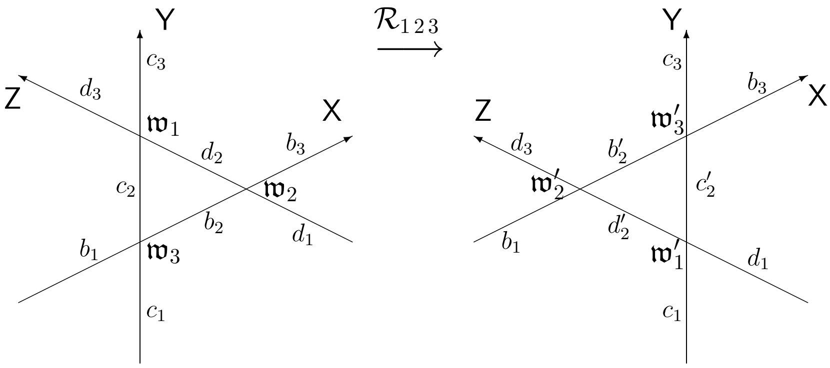

Figure 2. The canonical invertible mapping $\mathcal{R}_{123}$ shown in the auxiliary planes passing through the incoming (left) and outgoing (right) dynamic variables which are elements of $\mathfrak{w}_i$, respectively $\mathfrak{w}'_i$. The directed lines X, Y, Z are the intersections of the three planes forming the vertex A of figure 1. Their sections are labelled by the line-section parameters $b_1, \ldots, d_3$. Note that the choice of the orientation of the lines is not unique. The orientation chosen here corresponds to the numbering (44) and (45) of the fused vertex considered in section 3.

points $\left(p_1^{(j)}, p_2^{(j)}, p_3^{(j)}, p_4^{(j)}\right)$ each. These 16 points depend on five independent parameters expressed in terms of spherical angles, see [5]. Note that here on the left- and right-hand sides the same $p^{(j)}$ appear. This will not be the case in the generalizations which will be discussed soon.

Baxter and Forrester [19] have studied whether this model describes phase transitions. They used variational and numerical methods and found strong evidence that for the parameter values for which (10) is satisfied, the ZBB model is just at criticality [19]. So, in order to get a chance to describe phase transitions while staying integrable (recall that this is also a problem for the 2D Potts model), one should enlarge the framework and define more general Boltzmann weights and introduce less restrictive tetrahedron equations. Less restrictive and still powerful generalized equations can be used, as shown by Mangazeev and Stroganov [10]:

they introduced modified tetrahedron equations which guarantee commuting layer-to-layer transfer matrices. Further work along this line has been done in [11, 12].

1.2. **R**-*matrix satisfying the modified tetrahedron equation*

Since in the above-mentioned work [5] the proof that particular Boltzmann weights satisfy a particular MTE has been rather tedious, here we shall follow the approach introduced in [21] in which there is no need for an explicit check of the MTE. The Boltzmann weights are constructed from 'physical' principles which guarantee the validity of the MTE and nevertheless leave much freedom to obtain a broad class of integrable 3D models. We give a short summary of the argument.

One starts with an *oriented* 3D basic lattice. The dynamic variables living on the links i of this lattice are taken to be elements $u_i, w_i \in \mathfrak{w}_i$ an ultralocal Weyl algebra $\mathfrak{w} = \bigotimes \mathfrak{w}_i$ at the primitive Nth root of unity: $u_i w_j = q^{\delta_{ij}} w_j u_i$ (q as in equation (6)), which generalize the $\mathbb{Z}_N$-spin variables of (2) and (3). The Weyl elements are represented by standard $N \times N$ raising respectively diagonal matrices. The Nth powers of the Weyl variables are centres of the algebra and so are scalar variables.

The main object constructed is an invertible canonical mapping $\mathcal{R}_{ijk}$ in the space of a triple Weyl algebra. $\mathcal{R}_{ijk}$ operates at the vertices of the 3D lattice, mapping the three Weyl elements on the 'incoming' links onto those on the 'outgoing' links, see figure 1. The construction of $\mathcal{R}_{ijk}$ is based on two postulates, a Kirchhoff-like current conservation and a Baxter Z-invariance, and gives a unique explicit result: a canonical and invertible rational mapping operator. Since q is a root of unity, $\mathcal{R}_{ijk}$ decomposes into a matrix conjugation $\mathbf{R}_{ijk}$, and a purely functional mapping $\mathcal{R}^{(f)}_{ijk}$ which acts on the scalar parameters (the Weyl centres). So, for any rational function Φ on $\mathfrak{w}$:

$$\mathcal{R}_{123} \circ \Phi = \mathbf{R}_{123}\big(\mathcal{R}^{(f)}_{123} \circ \Phi\big)\mathbf{R}^{-1}_{123}. \tag{11}$$

It turns out that the matrix $\mathbf{R}_{ijk}$ has the form (7) where the four Fermat curve parameters, again constrained by (8), are rational functions of the scalar Weyl centre parameters.

Next consider an auxiliary plane which cuts the three incoming links near a vertex, and a second auxiliary plane cutting through the outgoing links, see figure 1. We take the six Weyl dynamic variables to sit on the six intersection points of the auxiliary planes. $\mathcal{R}_{123}$ can be regarded as the mapping of the ingoing auxiliary plane to the parallel shifted outgoing auxiliary plane.

Now consider the vertices of the basic lattice to be formed as the intersection points of three sets of non-parallel planes. The three planes which form the vertex A of figure 1 intersect the auxiliary planes in the lines $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ shown in figure 2. In figure 1 these intersection lines are the sides of the shaded triangles. Seen from the moving auxiliary plane, $\mathcal{R}_{123}$ shifts the line X through the vertex with index 1 or Y through the vertex 2 etc. We attach variables $b_1, b_2, \ldots, d_3$ to each section of the lines $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ as shown in figure 2.

It is convenient to parametrize the two scalar variables associated with the incoming dynamic Weyl variable $\mathfrak{w}_1$ (corresponding to u_1^N and w_1^N in the usual notation) by the ratios c_2^N / c_3^N and d_3^N / d_2^N. Analogously, e.g., those for $\mathfrak{w}'_2$ are defined as $b_1^N / b_2'^N$ and $d_2'^N / d_1^N$ etc. Details of the rule to parametrize the scalar variables in terms of 'line-section' variables $b_1, \ldots, d_3$ etc are explained in [9]. However, these will not be essential here, since one of the aims of this paper is to introduce and use another parametrization. Observe in figure 2 that $\mathcal{R}^{(f)}_{123}$ changes only three of the line-section parameters: b_2, c_2, d_2. From the explicit form of the canonical operator $\mathcal{R}_{123}$ (see [9]) one finds that the functional mapping $\mathcal{R}^{(f)}$ is rational in the *N*th powers of the line sections:

$$
\begin{aligned}
b_2'^N &= \frac{b_1^N c_3^N d_2^N + b_2^N c_3^N d_3^N + \kappa_1^N b_3^N c_2^N d_3^N}{c_2^N d_2^N} \\
c_2'^N &= \frac{\kappa_1^N b_3^N c_1^N d_2^N + \kappa_3^N b_2^N c_3^N d_1^N + \kappa_1^N \kappa_3^N b_3^N c_2^N d_1^N}{\kappa_2^N b_2^N d_2^N} \qquad (12) \\
d_2'^N &= \frac{b_2^N c_1^N d_3^N + b_1^N c_1^N d_2^N + \kappa_3^N b_1^N c_2^N d_1^N}{b_2^N c_2^N}.
\end{aligned}
$$

Here $\kappa_1, \kappa_2, \kappa_3$ are fixed parameters ('coupling constants') of the mapping $\mathcal{R}_{123}$. One can show [9] that in the line-section parametrization the three independent Fermat curve parameters which determine $\mathbf{R}_{123}$ according to (7) and (8) are

$$
x_1 = \frac{b_2 c_3}{\kappa_1 b_3 c_2} \qquad x_2 = \frac{\kappa_2 b_1 c_2'}{b_2' c_1} \qquad x_3 = \frac{b_1 c_3}{\sqrt{q} b_2' c_2}. \qquad (13)
$$

If we define a partition function in analogy to (2), the matrix elements of $\mathbf{R}_{ijk}$ take the role of generalized (because generically they will be complex) Boltzmann weights of integrable 3D lattice models of statistical mechanics. Despite their non-positivity we shall just call these matrix elements 'Boltzmann weights'.

Via the physical assumptions made in constructing $\mathcal{R}_{ijk}$, the validity of the TE is already built in. Simply considering two different sequences of Z-invariance shifts in a geometric figure formed by *four* intersecting straight lines ('quadrangle'), one concludes that (see, e.g., [9])

$$
\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} \sim \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123} \qquad (14)
$$

i.e. $\mathcal{R}_{ijk}$ satisfies the tetrahedron equation. Inserting (11) into (14) and choosing various phases of *N*th roots (the Fermat points (13) involve $b_1, \ldots,$ whereas the (12) relate $b_1^N, \ldots$) leads to the MTE for the matrix operator $\mathbf{R}_{ijk}$:

$$
\begin{aligned}
&\mathbf{R}_{123} \cdot \big(\mathcal{R}_{123}^{(f)} \circ \mathbf{R}_{145}\big) \cdot \big(\mathcal{R}_{123}^{(f)}\mathcal{R}_{145}^{(f)} \circ \mathbf{R}_{246}\big) \cdot \big(\mathcal{R}_{123}^{(f)}\mathcal{R}_{145}^{(f)}\mathcal{R}_{246}^{(f)} \circ \mathbf{R}_{356}\big) \\
&\quad \sim \mathbf{R}_{356} \cdot \big(\mathcal{R}_{356}^{(f)} \circ \mathbf{R}_{246}\big) \cdot \big(\mathcal{R}_{356}^{(f)}\mathcal{R}_{246}^{(f)} \circ \mathbf{R}_{145}\big) \cdot \big(\mathcal{R}_{356}^{(f)}\mathcal{R}_{246}^{(f)}\mathcal{R}_{145}^{(f)} \circ \mathbf{R}_{123}\big). \qquad (15)
\end{aligned}
$$

Via the Fermat points each $\mathbf{R}_{ijk}$ depends on several scalar variables, see, e.g., (13). In (15) the scalar variables which appear in the matrices $\mathbf{R}_{ijk}$ are to be transformed by the functional transformations $\mathcal{R}_{ijk}^{(f)}$ as indicated. Let us write shorthand

$$
\begin{aligned}
&\mathbf{R}^{(1)} = \mathbf{R}_{123} && \mathbf{R}^{(2)} = \mathcal{R}_{123}^{(f)} \circ \mathbf{R}_{145} && \mathbf{R}^{(3)} = \mathcal{R}_{123}^{(f)}\mathcal{R}_{145}^{(f)} \circ \mathbf{R}_{246} \\
&\mathbf{R}^{(4)} = \mathcal{R}_{123}^{(f)}\mathcal{R}_{145}^{(f)}\mathcal{R}_{246}^{(f)} \circ \mathbf{R}_{356} && \mathbf{R}^{(5)} = \mathcal{R}_{356}^{(f)}\mathcal{R}_{246}^{(f)}\mathcal{R}_{145}^{(f)} \circ \mathbf{R}_{123} && \\
&\mathbf{R}^{(6)} = \mathcal{R}_{356}^{(f)}\mathcal{R}_{246}^{(f)} \circ \mathbf{R}_{145} && \mathbf{R}^{(7)} = \mathcal{R}_{356}^{(f)} \circ \mathbf{R}_{246} && \mathbf{R}^{(8)} = \mathbf{R}_{356}.
\end{aligned} \qquad (16)
$$

Then the MTE (15) can be written compactly as

$$
\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}\mathbf{R}^{(4)} = \rho \mathbf{R}^{(8)}\mathbf{R}^{(7)}\mathbf{R}^{(6)}\mathbf{R}^{(5)} \qquad (17)
$$

where each $\mathbf{R}^{(j)}$ acts non-trivially in only three of the six spaces $\mathcal{V} = \mathbb{C}^N$. ρ is a scalar density factor which appears when passing from mappings to matrix equations.

The parameters which determine the $\mathbf{R}^{(j)}$ are the corresponding Fermat curve coordinates. Taking into account the functional transformations in (16) in terms of the line-section parameters one finds (for full details see [9]): $\mathbf{R}^{(j)} = \mathbf{R}\big(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}\big)$ with

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} \\ \vdots & \vdots & \vdots \\ x_1^{(7)} & x_2^{(7)} & x_3^{(7)} \\ x_1^{(8)} & x_2^{(8)} & x_3^{(8)} \end{pmatrix} = q^{-1/2} \begin{pmatrix} \frac{b_2 c_3}{\kappa_1 b_3 c_2} & \frac{\kappa_2 b_1 c_2'}{b_2' c_1} & \frac{b_1 c_3}{q^{1/2} b_2' c_2} \\ \frac{a_2 c_2'}{\kappa_1 a_3 c_1} & \frac{\kappa_4 a_1 c_1''}{a_2'' c_0} & \frac{a_1 c_2'}{q^{1/2} a_2'' c_1} \\ \vdots & \vdots & \vdots \\ \frac{a_2 b_3}{\kappa_2 a_3 b_2} & \frac{\kappa_4 a_1^{\dagger} b_2^{\dagger\dagger}}{a_2^{\dagger\dagger} b_1^{\dagger}} & \frac{a_1^{\dagger} b_3}{q^{1/2} a_2^{\dagger\dagger} b_2} \\ \frac{a_1 b_2}{\kappa_3 a_2 b_1} & \frac{\kappa_5 a_0 b_1^{\dagger}}{b_0 a_1^{\dagger}} & \frac{a_0 b_2}{q^{1/2} a_1^{\dagger} b_1} \end{pmatrix}. \tag{18}$$

The once or multiply transformed parameters such as $c_1'', b_1^{\dagger}$ follow from the iteration of equations such as (12). Altogether, since there are eight matrices $\mathbf{R}^{(j)}$ appearing in the MTEs, and as seen in figure 2, each transformation changes three line-section parameters, we have 24 equations for 32 different line-section parameters (these parameters can be seen in table 1). These form a set of classical integrable equations which are conveniently written in Hirota form:

$$\begin{aligned}
&b_2'^N c_2^N d_2^N = b_1^N c_3^N d_2^N + b_2^N c_3^N d_3^N + \kappa_1^N b_3^N c_2^N d_3^N \\
&\kappa_2^N b_2^N c_2'^N d_2^N = \kappa_1^N b_3^N c_1^N d_2^N + \kappa_3^N b_2^N c_3^N d_1^N + \kappa_1^N \kappa_3^N b_3^N c_2^N d_1^N \\
&b_2^N c_2^N d_2'^N = b_2^N c_1^N d_3^N + b_1^N c_1^N d_2^N + \kappa_3^N b_1^N c_2^N d_1^N \\[6pt]
&a_2''^N c_1^N d_1^N = a_1^N c_2'^N d_1^N + a_2^N c_2'^N d_2'^N + \kappa_1^N a_3^N c_1^N d_2'^N \\
&\kappa_4^N a_2^N c_1''^N d_1^N = \kappa_1^N a_3^N c_0^N d_1^N + \kappa_5^N a_2^N c_2'^N d_0^N + \kappa_1^N \kappa_5^N a_3^N c_1^N d_0^N \\
&a_2^N c_1^N d_1''^N = a_2^N c_0^N d_2'^N + a_1^N c_0^N d_1^N + \kappa_5^N a_1^N c_1^N d_0^N \\[6pt]
&a_1'''^N b_1^N d_2'^N = a_0^N b_2'^N d_2'^N + d_3^N a_1^N b_2'^N + \kappa_2^N d_3^N b_1^N a_2''^N \\
&\kappa_4^N a_1^N b_1'''^N d_2'^N = \kappa_2^N b_0^N a_2''^N d_2'^N + \kappa_6^N a_1^N b_2'^N d_1''^N + \kappa_2^N \kappa_6^N a_2''^N b_1^N d_1''^N \\
&a_1^N b_1^N d_2'''^N = b_0^N d_3^N a_1^N + a_0^N b_0^N d_2'^N + \kappa_6^N a_0^N b_1^N d_1''^N \\[6pt]
&a_2^{\dagger\dagger N} b_2'^N c_2'^N = b_3^N a_1'''^N c_2'^N + b_3^N c_3^N a_2''^N + \kappa_3^N a_3^N c_3^N b_2'^N \\
&\kappa_5^N a_2''^N b_2^{\dagger\dagger N} c_2'^N = \kappa_3^N a_3^N b_1'''^N c_2'^N + \kappa_6^N b_3^N a_2''^N c_1''^N + \kappa_3^N \kappa_6^N a_3^N b_2'^N c_1''^N \\
&a_2''^N b_2'^N c_2^{\dagger\dagger\dagger N} = c_3^N a_2''^N b_1'''^N + a_1'''^N b_1'''^N c_2'^N + \kappa_6^N a_1'''^N b_2'^N c_1''^N \\[6pt]
&\vdots \qquad\qquad\qquad \vdots \\[6pt]
&b_1'''^N c_1^{\dagger N} d_1^{\dagger\dagger N} = b_0^N c_2^{\dagger\dagger\dagger N} d_1^{\dagger\dagger N} + b_1^{\dagger N} c_2^{\dagger\dagger\dagger N} d_2^{\dagger\dagger\dagger N} + \kappa_1^N b_2^{\dagger\dagger N} c_1^{\dagger N} d_2^{\dagger\dagger\dagger N} \\
&\kappa_2^N b_1^{\dagger N} c_1''^N d_1^{\dagger\dagger N} = \kappa_1^N c_0^N b_2^{\dagger\dagger N} d_1^{\dagger\dagger N} + \kappa_3^N d_0^N b_1^{\dagger N} c_2^{\dagger\dagger\dagger N} + \kappa_1^N \kappa_3^N d_0^N b_2^{\dagger\dagger N} c_1^{\dagger N} \\
&b_1^{\dagger N} c_1^{\dagger N} d_1''^N = c_0^N b_1^{\dagger N} d_2^{\dagger\dagger\dagger N} + b_0^N c_0^N d_1^{\dagger\dagger N} + \kappa_3^N b_0^N d_0^N c_1^{\dagger N}.
\end{aligned} \tag{19}$$

The first three of these equations are just (12), defining $\mathcal{R}_{123}^{(f)}$, i.e. $b_2'^N, c_2'^N, d_2'^N$, in terms of the unprimed $b_1, \ldots, d_3$. The first six equations together (e.g., expressed in the fourth equation on the right-hand side $c_2'^N$ and $d_2'^N$ from the first and third equations) define $\mathcal{R}^{(f)}_{123} \circ \mathcal{R}^{(f)}_{145}$, etc. The complete expressions for (18) and (19) can be found in [9].

Straightforward combination of the first 12 equations of (19) on one hand, and of the last 12 equations of (19) on the other hand (best done, e.g., by Maple), shows that the functional mappings given in (19) automatically satisfy the functional TE:

$$\mathcal{R}^{(f)}_{123} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{356} = \mathcal{R}^{(f)}_{356} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{123} \qquad (20)$$

where for the superposition of two operators acting on a function Φ we use the notation $((\mathcal{A} \cdot \mathcal{B}) \cdot \Phi) \stackrel{\text{def}}{=} (\mathcal{A} \cdot (\mathcal{B} \cdot \Phi))$. Of course, the validity of (20) is a consequence of the physical rules used when constructing $\mathcal{R}_{ijk}$. In the line-section parametrization the relation between the first, second etc lines in both (18) and (19) is not transparent. Introducing a new parametrization in the next subsection will make these relations simple and explicit.

2. Parametrization using concepts of algebraic geometry

2.1. Theta functions

It is well known [14–18] that Hirota-type equations can be identically satisfied by a parametrization in terms of theta functions on an algebraic curve. We shall now introduce such a parametrization in order to write (18) and (19) in a more systematic way. This will be also useful later to formulate fusion in a transparent manner. For the notation of algebraic geometry see, e.g., [13].

Let Γ_g be an abstract generic algebraic curve of the genus g with $\boldsymbol{\omega}$ being the canonical g-dimensional vector of the homomorphic differentials. For any two points $X, Y \in \Gamma_g$ let $\mathbf{I}^X_Y : \Gamma_g^2 \mapsto \mathrm{Jac}(\Gamma_g)$ be

$$\mathbf{I}^X_Y \stackrel{\text{def}}{=} \int_Y^X \boldsymbol{\omega}. \qquad (21)$$

Let further $E(X, Y) = -E(Y, X)$ be the prime form on Γ_g^2, and $\Theta(\mathbf{v})$ be the theta function on $\mathrm{Jac}(\Gamma_g)$.

It is well known that the theta functions on the Jacobian of an algebraic curve obey the Fay identity

$$\begin{aligned}\Theta(\mathbf{v})\Theta\big(\mathbf{v} + \mathbf{I}^A_B + \mathbf{I}^C_D\big) &= \Theta\big(\mathbf{v} + \mathbf{I}^A_D\big)\Theta\big(\mathbf{v} + \mathbf{I}^C_B\big)\frac{E(A, B)E(D, C)}{E(A, C)E(D, B)} \\ &\quad + \Theta\big(\mathbf{v} + \mathbf{I}^A_B\big)\Theta\big(\mathbf{v} + \mathbf{I}^C_D\big)\frac{E(A, D)E(C, B)}{E(A, C)E(D, B)}\end{aligned} \qquad (22)$$

which involves four points $A, B, C, D \in \Gamma_g$ and a $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$. We shall show that in the parametrization to be introduced below, the Fermat relations become just Fay identities. The Fay identity involves only cross-ratios of prime forms, and these ratios have a simple expression in terms of non-singular odd characteristic theta functions:

$$\begin{bmatrix} X & X' \\ Y & Y' \end{bmatrix} \stackrel{\text{def}}{=} \frac{E(X, Y)E(X', Y')}{E(X, Y')E(X', Y)} = \frac{\Theta_{\epsilon_{\text{odd}}}\big(\mathbf{I}^Y_X\big)\Theta_{\epsilon_{\text{odd}}}\big(\mathbf{I}^{Y'}_{X'}\big)}{\Theta_{\epsilon_{\text{odd}}}\big(\mathbf{I}^{Y'}_X\big)\Theta_{\epsilon_{\text{odd}}}\big(\mathbf{I}^Y_{X'}\big)}. \qquad (23)$$

For solving the 24 trilinear equations (19) we shall need an identity with more arguments $Q, X, Y, Y', Z, Z' \in \Gamma_g$ obtained by combining two Fay identities:

$$\Theta\big(\mathbf{v}+\mathbf{I}_X^Q\big)\Theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_Z^{Z'}\big)\Theta\big(\mathbf{v}+\mathbf{I}_Z^Q+\mathbf{I}_Y^{Y'}\big)$$
$$-\Theta\big(\mathbf{v}+\mathbf{I}_X^Q+\mathbf{I}_Z^{Z'}\big)\Theta\big(\mathbf{v}+\mathbf{I}_Y^Q\big)\Theta\big(\mathbf{v}+\mathbf{I}_Z^Q+\mathbf{I}_Y^{Y'}\big)\begin{bmatrix}X & Y\\ Z & Z'\end{bmatrix}$$
$$-\Theta\big(\mathbf{v}+\mathbf{I}_X^Q+\mathbf{I}_Y^{Y'}\big)\Theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_Z^{Z'}\big)\Theta\big(\mathbf{v}+\mathbf{I}_Z^Q\big)\begin{bmatrix}X & Z\\ Y & Y'\end{bmatrix}$$
$$+\Theta\big(\mathbf{v}+\mathbf{I}_X^Q+\mathbf{I}_Y^{Y'}+\mathbf{I}_Z^{Z'}\big)\Theta\big(\mathbf{v}+\mathbf{I}_Y^Q\big)\Theta\big(\mathbf{v}+\mathbf{I}_Z^Q\big)\begin{bmatrix}X & Y\\ Z & Z'\end{bmatrix}\begin{bmatrix}X & Z'\\ Y & Y'\end{bmatrix}=0. \tag{24}$$

Furthermore, since we will need the Nth roots of theta functions and prime forms, we also define $e(X,Y)$ and $\theta(\mathbf{v})$ by

$$e(X,Y)^N=\Theta_{\epsilon_{\rm odd}}\big(\mathbf{I}_Y^X\big)\sim E(X,Y)\qquad \theta(\mathbf{v})^N=\Theta(\mathbf{v}). \tag{25}$$

Since in the following we shall have to write many equations involving theta functions, it is convenient to introduce special abbreviations. For $Q, A, B_1, B_1', \ldots \in \Gamma_g$ we define:

$$(A, B_1+B_2+\cdots+B_n)\equiv\Theta\left(\mathbf{v}+\mathbf{I}_A^Q+\sum_{j=1}^n \mathbf{I}_{B_j}^{B_j'}\right)\qquad \langle A,B\rangle\equiv E(A,B)$$
$$[A, B_1+B_2+\cdots+B_n]\equiv\theta\left(\mathbf{v}+\mathbf{I}_A^Q+\sum_{j=1}^n \mathbf{I}_{B_j}^{B_j'}\right) \tag{26}$$
$$\{A,B\}\equiv -q^{-1/2}e(A,B')/e(A,B)\qquad \overline{\{A,B\}}\equiv -e(A',B)/e(A,B).$$

Note that the brackets (,) and [,] introduced here do not show explicitly the dependence on the variables $\mathbf{v}$, Q, $B_1', \ldots, B_n'$ since these always come in the same form. We also introduce, using this notation:

$$\begin{aligned}\mathbf{F}(\mathbf{v};X,Y',Y,Z',Z)\stackrel{\rm def}{=}{}&(X)(Y,Z)(Z,Y)\langle Y,Z\rangle\langle X,Z'\rangle\langle X,Y'\rangle\\ &-(X,Z)(Y)(Z,Y)\langle X,Z\rangle\langle Y,Z'\rangle\langle X,Y'\rangle\\ &-(X,Y)(Y,Z)(Z)\langle X,Y\rangle\langle Y',Z\rangle\langle X,Z'\rangle\\ &+(X,Y+Z)(Y)(Z)\langle X,Z\rangle\langle X,Y\rangle\langle Y',Z'\rangle\end{aligned} \tag{27}$$

so that the double-Fay identity (24) is

$$\mathbf{F}(\mathbf{v};X,Y',Y,Z',Z)=0. \tag{28}$$

The dependence on Q is trivial since it appears only in the combination $\mathbf{v}+\mathbf{I}_{\ldots}^Q$. So Q is not an independent variable.

2.2. *Re-parametrization of* R

Let us introduce the new parametrization of the matrix (7). As we illustrated in figure 2, in the auxiliary plane the mapping $\mathcal{R}_{123}$ can be considered as a relative shift of three directed lines X, Y, Z with respect to each other. Now, for the given algebraic curve Γ_g and $\mathbf{v}\in\mathbf{C}^g$, we introduce three pairs of points on Γ_g:

$$X', X, Y', Y, Z', Z\in\Gamma_g. \tag{29}$$

Another point $Q\in\Gamma_g$ will just serve as a trivial normalization. Then let

$$\mathbf{R}=\mathbf{R}(p_1,p_2,p_3,p_4)\quad\Longleftrightarrow\quad \mathbf{R}=\mathbf{R}(\mathbf{v};X',X;Y',Y;Z',Z) \tag{30}$$

with, using the shorthand notations (26) and $p_j = (x_j, y_j)$,

$$
\begin{aligned}
x_1 &= \frac{1}{q}\frac{\{X,Z\}}{\{Y',Z\}}\frac{[X,Y][Y,Z]}{[X,Y+Z][Y]} & y_1 &= \frac{e(Z,Z')e(X,Y')}{e(X,Z)e(Y',Z')}\frac{[Z,Y]\theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'}\big)}{[X,Y+Z][Y]}\\
x_2 &= \frac{\{X',Z\}}{\{Y,Z\}}\frac{[X][Y,X+Z]}{[X,Z][Y,X]} & y_2 &= q\frac{e(Z,Z')e(X',Y)}{e(X',Z)e(Y,Z')}\frac{[Z,X]\theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'}\big)}{[X,Z][Y,X]}\\
x_3 &= \frac{1}{q}\frac{\{X,Z\}}{\{Y,Z\}}\frac{[X][Y,Z]}{[X,Z][Y]} & y_3 &= q\frac{e(Z,Z')e(X,Y)}{e(X,Z)e(Y,Z')}\frac{[Z]\theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'}\big)}{[X,Z][Y]}\\
x_4 &= \frac{1}{q}\frac{\{X',Z\}}{\{Y',Z\}}\frac{[X,Y][Y,X+Z]}{[X,Y+Z][Y,X]} & y_4 &= \frac{e(Z,Z')e(X',Y')}{e(X',Z)e(Y',Z')}\frac{[Z,X+Y]\theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'}\big)}{[X,Y+Z][Y,X]}.
\end{aligned}
\tag{31}
$$

Actually, see (5), for defining $\mathbf{R}_{123}$ we do not need the y_i themselves but only the ratios

$$
\begin{aligned}
\frac{y_3}{y_1} &= q\frac{\{\overline{Y,Z'}\}}{\{X,Y\}}\frac{[Z][X,Y+Z]}{[X,Z][Z,Y]}\\
\frac{y_4}{y_1} &= \frac{\{\overline{X,Y'}\}}{\{\overline{X,Z}\}}\frac{[Z,X+Y][Y]}{[Z,Y][Y,X]}\\
\frac{y_3}{y_2} &= \frac{\{\overline{X,Z}\}}{\{\overline{X,Y}\}}\frac{[Z][Y,X]}{[Y][Z,X]}
\end{aligned}
\tag{32}
$$

from which $e(Z,Z')$ and $\theta\big(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'}\big)$ drop out.

Note that for this parametrization we used a generic algebraic curve and generic points on this curve, and a generic point on its Jacobian, all in order to parametrize just three independent complex numbers x_1, x_2, x_3. In (31) all x_k, y_k are the periodical functions of $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$.

The parametrization (31) is suggested by a few assumptions: first, the prime forms shall appear in the x_i only in the form of Nth roots of (23):

$$
\frac{\{X,Z\}}{\{Y,Z\}} = \begin{bmatrix} X & Y \\ Z & Z' \end{bmatrix}^{1/N}. \tag{33}
$$

Second, considering (13), we demand that the line-section parameters b_1, b_2, b_3, b_2' (sections of the line X in figure 2) should be proportional to Nth roots of theta functions of the form $[X, \ldots]$ defined in (26). Analogously, the sections $c_1, \ldots, c_2'$ of the line Y are assumed to be proportional to $[Y, \ldots]$. Finally, we consider that we want to use Fay identities to provide the Fermat relations and the Hirota equations.

The merit of this parametrization will be seen in several places: when we consider the transformed mappings $\mathbf{R}^{(2)}, \ldots, \mathbf{R}^{(6)}$, when we re-write equations (19) and when we construct composite weights in section 3.

We must now verify that (31), and its generalization to the other Fermat points in the MTE, give a consistent parametrization of the relevant equations (4), (18) and (19). We first check that (31) satisfies the Fermat relations

$$
x_j^N + y_j^N = 1. \tag{34}
$$

Indeed, these are true due to the Fay identity, which for $A, B, C, D \in \Gamma_g$ we write as

$$
\begin{aligned}
-\langle A,C\rangle\langle D,B\rangle\Theta(\mathbf{v})\Theta\big(\mathbf{v}+\mathbf{I}_B^A+\mathbf{I}_D^C\big) + \langle A,B\rangle\langle D,C\rangle\Theta\big(\mathbf{v}+\mathbf{I}_D^A\big)\Theta\big(\mathbf{v}+\mathbf{I}_B^C\big)\\
+\langle A,D\rangle\langle C,B\rangle\Theta\big(\mathbf{v}+\mathbf{I}_B^A\big)\Theta\big(\mathbf{v}+\mathbf{I}_D^C\big) = 0.
\end{aligned}
\tag{35}
$$

For $j = 1$ put in (35) $(A,B,C,D) \to (Y',X,Z',Z)$ and $\mathbf{v} \to \mathbf{v}' = \mathbf{v} + \mathbf{I}_Y^Q$, giving

$$
\begin{aligned}
\langle Z,X\rangle\langle Y',Z'\rangle(Y)(X,Y+Z) - \langle Z,Z'\rangle\langle Y',X\rangle(Z,Y)\Theta\big(\mathbf{v}'+\mathbf{I}_X^{Z'}\big)\\
- \langle Z',X\rangle\langle Y',Z\rangle(X,Y)(Y,Z) = 0
\end{aligned}
$$

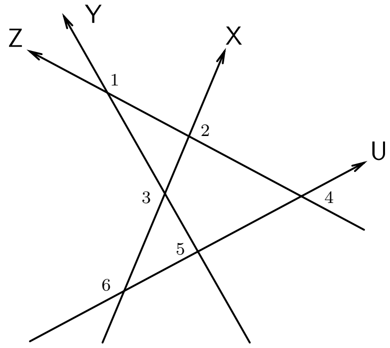

Figure 3. Quadrangle in the auxiliary plane formed by the directed intersection lines of four oriented lattice planes. The six spaces $\mathcal{V}$ in which the MTE operates are considered to be located at the six intersection points.

for $j = 2$ put in (35) $(A, B, C, D) \to (X', Y, Z', Z)$ and $\mathbf{v} \to \mathbf{v}'' = \mathbf{v} + \mathbf{I}_X^Q$, giving

$$\langle Z', Y\rangle\langle X', Z\rangle(Y, X)(X, Z) - \langle Z, Y\rangle\langle X', Z'\rangle(X)(Y, X + Z) + \langle Z, Z'\rangle\langle X', Y\rangle(Z, X)\Theta\big(\mathbf{v}'' + \mathbf{I}_Y^{Z'}\big) = 0$$

for $j = 3$ put in (35) $(A, B, C, D) \to (Z, X, Z', Y)$ and $\mathbf{v} \to \mathbf{v}^+ = \mathbf{v} + \mathbf{I}_Z^Q$:

$$\langle X, Z'\rangle\langle Y, Z\rangle(X)(Y, Z) - \langle X, Z\rangle\langle Y, Z'\rangle(Y)(X, Z) + \langle Z, Z'\rangle\langle X, Y\rangle(Z)\Theta\big(\mathbf{v} + \mathbf{I}_Y^Q + \mathbf{I}_X^{Z'}\big) = 0$$

or $\langle X, Z'\rangle\langle Y, Z\rangle\Theta\big(\mathbf{v}^+ + \mathbf{I}_X^Z\big)\Theta\big(\mathbf{v}^+ + \mathbf{I}_Y^{Z'}\big) - \langle X, Z\rangle\langle Y, Z'\rangle\Theta\big(\mathbf{v}^+ + \mathbf{I}_Y^Z\big)\Theta\big(\mathbf{v}^+ + \mathbf{I}_X^{Z'}\big)$

$$+ \langle Z, Z'\rangle\langle X, Y\rangle\Theta(\mathbf{v}^+)\Theta\big(\mathbf{v}^+ + \mathbf{I}_X^Z + \mathbf{I}_Y^{Z'}\big) = 0.$$

For $j = 4$ put in (35) $(A, B, C, D) \to (Y', Z, X', Z')$, $\mathbf{v} \to \mathbf{v}^* = \mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Y^{Z'}$.

2.3. Line-section parameters and Hirota equations in terms of theta functions

For writing the MTE in our new parametrization and to check (18) and (19), we consider three more spaces $\mathcal{V} = \mathbb{C}^N$, corresponding to the indices 4, 5, 6. In figure 2 the first three spaces were located at the intersection points of the lines X, Y, Z. To include the other three spaces, consider the 'quadrangle' formed by *four* lines shown in figure 3. Corresponding to the new line U we introduce another pair of points $U', U \in \Gamma_g$.

Looking at figure 3 we see that, instead of labelling the spaces by the vertices $1, \ldots, 6$ of the quadrangle, we can also label them by the pair of lines which intersect in these vertices, so identifying

$$1 \sim \mathsf{YZ} \qquad 2 \sim \mathsf{XZ} \qquad 3 \sim \mathsf{XY} \qquad 4 \sim \mathsf{UZ} \qquad 5 \sim \mathsf{UY} \qquad 6 \sim \mathsf{UX}. \tag{36}$$

Note that the ordering of the lines is important for the identification (36): we shall choose the anti-clockwise orientation in figure 3, not the mirror reflected clockwise orientation.

Next we assume that we can write the 'coupling constants' κ_j all in the form (33). Then from (36) it is suggested to build, e.g., κ_1 from the points Y', Y, Z', Z only, etc and put (factors $q^{1/2}$ are inserted to produce correct signs when forming Nth powers for (19)):

$$\kappa_1 = q^{1/2}\frac{\{Y', Z\}}{\{Y, Z\}} \qquad \kappa_2 = q^{1/2}\frac{\{X', Z\}}{\{X, Z\}} \qquad \kappa_3 = q^{1/2}\frac{\{X', Y\}}{\{X, Y\}}$$
$$\kappa_4 = q^{1/2}\frac{\{U', Z\}}{\{U, Z\}} \qquad \kappa_5 = q^{1/2}\frac{\{U', Y\}}{\{U, Y\}} \qquad \kappa_6 = q^{1/2}\frac{\{U', X\}}{\{U, X\}}. \tag{37}$$

Table 1. The 32 line-section parameters appearing in equations (19), expressed in terms of theta functions and prime factor ratios, using the abbreviations (26). Observe that in the prime factor brackets { , } the points come always in the order U, X, Y, Z (without and with primes).

$a_0 = [U]\{U, X\}\{U, Y\}\{U, Z\}$	$b_1 = [X]\{X, Y\}\{X, Z\}\{\overline{U, X}\}$
$a_1 = [U, X]\{U, Y\}\{U, Z\}$	$b_2 = [X, Y]\{X, Z\}\{\overline{U, X}\}$
$a_1^{\dagger} = [U, Y]\{U, Z\}\{U, X\}$	$b_2' = [X, Z]\{X, Y\}\{\overline{U, X}\}$
$a_1''' = a_1^{\dagger\dagger\dagger} = [U, Z]\{U, X\}\{U, Y\}$	$b_0 = [X, U]\{X, Y\}\{X, Z\}$
$a_2 = [U, X + Y]\{U, Z\}$	$b_3 = [X, Y + Z]\{\overline{U, X}\}$
$a_2^{\dagger\dagger} = a_2^T = [U, Y + Z]\{U, X\}$	$b_1''' = b_1^t = [X, U + Z]\{X, Y\}$
$a_2'' = [U, X + Z]\{U, Y\}$	$b_1^{\dagger} = [X, U + Y]\{X, Z\}$
$a_3 = [U, X + Y + Z]$	$b_2^{\dagger\dagger} = b_2^T = [X, U + Y + Z]$
$c_2 = [Y]\{Y, Z\}\{\overline{U, Y}\}\{\overline{X, Y}\}$	$d_3 = [Z]\{\overline{U, Z}\}\{\overline{X, Z}\}\{\overline{Y, Z}\}$
$c_1^{\dagger} = [Y, U]\{Y, Z\}\{\overline{X, Y}\}$	$d_2''' = d_2^{\dagger\dagger\dagger} = [Z, U]\{\overline{X, Z}\}\{\overline{Y, Z}\}$
$c_3 = [Y, Z]\{\overline{U, Y}\}\{\overline{X, Y}\}$	$d_2' = [Z, X]\{\overline{Y, Z}\}\{\overline{U, Z}\}$
$c_1 = [Y, X]\{Y, Z\}\{\overline{U, Y}\}$	$d_2 = [Z, Y]\{\overline{U, Z}\}\{\overline{X, Z}\}$
$c_0 = [Y, U + X]\{Y, Z\}$	$d_1'' = d_1^t = [Z, U + X]\{\overline{Y, Z}\}$
$c_2^{\dagger\dagger\dagger} = c_2^T = [Y, Z + U]\{\overline{X, Y}\}$	$d_1 = [Z, X + Y]\{\overline{U, Z}\}$
$c_2' = [Y, X + Z]\{\overline{U, Y}\}$	$d_1^{\dagger\dagger} = [Z, U + Y]\{\overline{X, Z}\}$
$c_1'' = c_1^t = [Y, Z + U + X]$	$d_0 = [Z, U + X + Y]$

Now we consider (31) and (37) and assume that the line-section parameters $a_0, a_1, \ldots$ and $d_0, d_1, \ldots$ follow the same scheme as postulated for $b_0, \ldots, c_0, \ldots$ above after (33): $a_i \sim [U, \ldots]$, $d_i \sim [Z, \ldots]$. So equations (18) lead us to express all line-section parameters in terms of theta functions as shown in table 1. We make ample use of the short-hand notation (26).

Apart from Q which always comes with $\mathbf{v}$, we use eight arbitrary points $X', X, Y', Y, Z', Z, U', U \in \Gamma_g$. The κ_j may also be written in terms of the bare brackets using $\{\overline{A, B'}\}\{A, B\} = \{\overline{A, B}\}\{A', B\}$. From table 1 we see that the a_i do not depend on U', the b_i do not depend on X' etc.

Now, using the results of table 1 and (37), we shall re-write all the Hirota equations (19) in terms of theta functions on Γ_g and prime form cross-ratios. Not very surprisingly in view of [14, 17, 18], it turns out that all these have the form of the double-Fay identity. Also, as expected from figure 3, and the meaning of the mappings as moving lines within the quadrangle, the 24 equations follow from each other by a sequence of simple substitutions. Inserting from table 1 and (37), the first three equations of (19) become, using the notation (27) (recall that these are equations (12) defining the functional mapping $\mathcal{R}_{123}^{(f)}$):

$$\begin{aligned} &\mathbf{F}(\mathbf{v}; X, Y', Y, Z', Z)\frac{(\{\overline{U, X}\}\{\overline{U, Y}\}\{\overline{U, Z}\}\{\overline{X, Y}\}\{\overline{X, Z}\})^N}{E(X, Y)E(X, Z)E(Y, Z)} = 0 \\ &\mathbf{F}(\mathbf{v} + \mathbf{I}_Y^{Y'}; Y', X', X, Z', Z)\frac{(\{\overline{U, X}\}\{\overline{U, Y}\}\{\overline{U, Z}\})^N}{E(X, Z)E(X, Y')E(Y', Z)} = 0 \\ &\mathbf{F}(\mathbf{v}; Z, Y', Y, X', X)\frac{(\{\overline{U, X}\}\{\overline{U, Y}\}\{\overline{U, Z}\}\{X, Z\}\{Y, Z\})^N}{E(X, Y)E(X, Z)E(Y, Z)} = 0. \end{aligned} \tag{38}$$

The dependence on U', U appears only in the factors on the right, not in the $\mathbf{F}$. Assuming generic points $U', U, \ldots$ we conclude that the $\mathbf{F}$ must vanish and we combine the essential terms of (38) into

$$\mathfrak{F}(\mathbf{v}; X', X, Y', Y, Z', Z) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{F}(\mathbf{v}; X, Y', Y, Z', Z) \\ \mathbf{F}(\mathbf{v} + \mathbf{I}_Y^{Y'}; Y', X', X, Z', Z) \\ \mathbf{F}(\mathbf{v}; Z, Y', Y, X', X) \end{pmatrix}. \tag{39}$$

Then the 24 Hirota equations (19) which describe the functional mappings take the form

$$\begin{array}{ll} \mathfrak{F}(\mathbf{v}; X', X, Y', Y, Z', Z) = 0 & \mathfrak{F}\big(\mathbf{v} + \mathbf{I}_U^{U'}; X', X, Y', Y, Z', Z\big) = 0 \\ \mathfrak{F}\big(\mathbf{v} + \mathbf{I}_X^{X'}; U', U, Y', Y, Z', Z\big) = 0 & \mathfrak{F}(\mathbf{v}; U', U, Y', Y, Z', Z) = 0 \\ \mathfrak{F}(\mathbf{v}; U', U, X', X, Z', Z) = 0 & \mathfrak{F}\big(\mathbf{v} + \mathbf{I}_Y^{Y'}; U', U, X', X, Z', Z\big) = 0 \\ \mathfrak{F}\big(\mathbf{v} + \mathbf{I}_Z^{Z'}; U', U, X', X, Y', Y\big) = 0 & \mathfrak{F}(\mathbf{v}; U', U, X', X, Y', Y) = 0. \end{array} \qquad (40)$$

The equations in the left column of (40) precisely correspond to the first 12 equations of (19). The last three equations of (19) are combined into the top equation of the right column of (40). As already mentioned with (20), equations (40) together contain the functional TE.

2.4. Theta-parametrization of the simple modified tetrahedron equation

Finally, we use the parametrization (31) to re-write the MTE. From (17) with (18) we find that the functional mapping just produces a permutation of the four pairs of points $X, X', \ldots, U, U' \in \Gamma_g$, together with shifts in the vector $\mathbf{v}$. Of course, the result corresponds to (40). Explicitly it is:

Theorem 1. *The simple modified tetrahedron equation may be parametrized in terms of* Γ_g, $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$ *and four pairs* $X', X, Y', Y, Z', Z, U', U \in \Gamma_g$ *by definitions (30), (31), (18) as follows:*

$$\begin{array}{ll} \mathbf{R}^{(1)} = \mathbf{R}(\mathbf{v}; X', X; Y', Y; Z', Z) & \mathbf{R}^{(5)} = \mathbf{R}\big(\mathbf{v} + \mathbf{I}_U^{U'}; X', X; Y', Y; Z', Z\big) \\ \mathbf{R}^{(2)} = \mathbf{R}\big(\mathbf{v} + \mathbf{I}_X^{X'}; U', U; Y', Y; Z', Z\big) & \mathbf{R}^{(6)} = \mathbf{R}(\mathbf{v}; U', U; Y', Y; Z', Z) \\ \mathbf{R}^{(3)} = \mathbf{R}(\mathbf{v}; U', U; X', X; Z', Z) & \mathbf{R}^{(7)} = \mathbf{R}\big(\mathbf{v} + \mathbf{I}_Y^{Y'}; U', U; X', X; Z', Z\big) \\ \mathbf{R}^{(4)} = \mathbf{R}\big(\mathbf{v} + \mathbf{I}_Z^{Z'}; U', U; X', X; Y', Y\big) & \mathbf{R}^{(8)} = \mathbf{R}(\mathbf{v}; U', U; X', X; Y', Y). \end{array} \qquad (41)$$

Proof. Each $\mathbf{R}^{(j)}$ is determined by its three Fermat points $x_1^{(j)}, x_2^{(j)}, x_3^{(j)}$. From [9] these points are known in terms of the line-section parameters, see (18). Inserting the theta-function expressions for the line sections from table 1 into (18) one finds that the $x_i^{(j)}$ for $j = 2, \ldots, 8$ are obtained from those for $j = 1$, equations (31), by the substitutions seen in (41). □

Using the correspondence between the labels $1, \ldots, 6$ and the line labels $\mathsf{U}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}$, given in (36), $\mathbf{R}^{(1)} = \mathbf{R}_{123}$ may also be labelled as $\mathbf{R}^{\mathsf{XYZ}}$ etc, and we write the MTE as

$$\begin{aligned} &\mathbf{R}^{\mathsf{XYZ}}(\mathbf{v})\mathbf{R}^{\mathsf{UYZ}}\big(\mathbf{v} + \mathbf{I}_X^{X'}\big)\mathbf{R}^{\mathsf{UXZ}}(\mathbf{v})\mathbf{R}^{\mathsf{UXY}}\big(\mathbf{v} + \mathbf{I}_Z^{Z'}\big) \\ &\qquad = \rho\mathbf{R}^{\mathsf{UXY}}(\mathbf{v})\mathbf{R}^{\mathsf{UXZ}}\big(\mathbf{v} + \mathbf{I}_Y^{Y'}\big)\mathbf{R}^{\mathsf{UYZ}}(\mathbf{v})\mathbf{R}^{\mathsf{XYZ}}\big(\mathbf{v} + \mathbf{I}_U^{U'}\big). \end{aligned} \qquad (42)$$

This notation also indicates directly the three pairs of points on the algebraic curve which parametrize the matrices $\mathbf{R}^{(j)}$ in (41).

In [9] we showed that using simple re-scalings, out of the 24 line-section parameters listed in table 1 and the six parameters $\kappa_1, \ldots, \kappa_6$, only eight parameters are independent. Here we have eight points on Γ_g which can be chosen freely. In addition, 16 phases from taking the *N*th roots can be chosen freely. In terms of the line-section parameters, the choice of the independent phases is the same as that explained in [9].

3. The fused vertex weight $\mathfrak{R}$

3.1. Open $N_1 \times N_2 \times N_3$ box

The natural graphical interpretation of the **R**-matrix is a three-dimensional vertex, i.e. the intersection of three planes in 3D space. The six indices σ_j, σ_j' are associated with the edges of the vertex, recall figure 1.

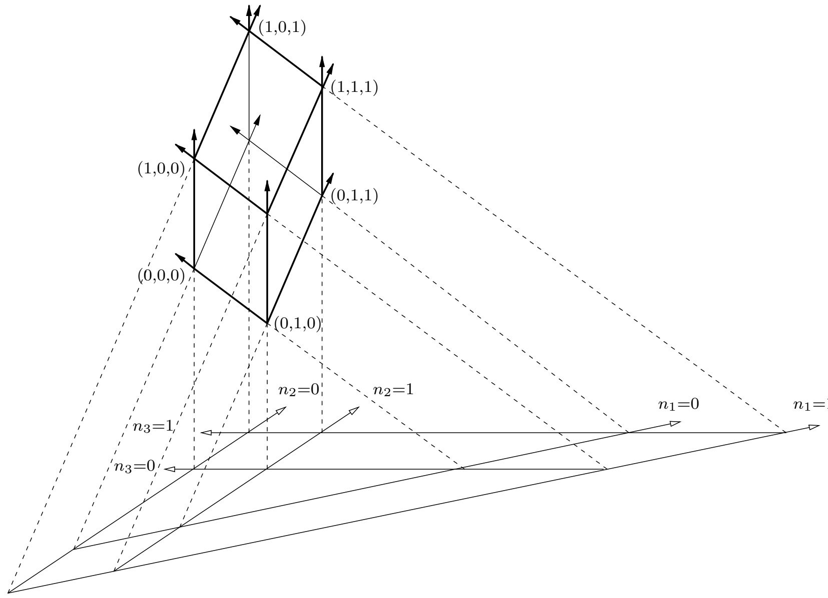

Figure 4. Top: three-dimensional view of the oriented cube $N_1 = N_2 = N_3 = 2$ (heavy lines) which is formed by six planes (indicated by dashed lines). Bottom: the horizontal auxiliary plane with the three pairs of lines arising from the intersection of the three pairs of planes with the auxiliary plane. This is a generalization of figures 1 and 2: if we consider, e.g., the point (0, 1, 0) to correspond to the point *A* of figure 1, then the inner triangle in the auxiliary plane corresponds to the left shaded triangle of figure 1 and to the left part of figure 2. So the initial external spin (Weyl) variables (44) can be considered to sit at the three times four intersection points of the auxiliary plane. In order to get a similar picture for the final external variables we have to place the auxiliary plane above the cube.

The next step is the consideration of the intersection of three *sets* of N_1, N_2 and N_3 parallel planes. This produces a finite open cubic lattice of the size $N_1 \times N_2 \times N_3$. We call the corresponding vertex object $\mathfrak{R}$. It is the result of the fusion of elementary **R**-matrices.

The lattice is defined as in (1), but now $n_j = 0, \ldots N_j - 1$. Let $\mathfrak{R}_{123}$ be the matrix associated with the open cube (more precisely, the open parallelepiped):

$$\mathfrak{R}_{123} \equiv \langle \vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3 | \mathfrak{R} | \vec{\sigma}_1', \vec{\sigma}_2', \vec{\sigma}_3' \rangle = \sum_{\{\sigma\}} \prod_{\mathbf{n}} \big\langle \sigma_{1,\mathbf{n}}, \sigma_{2,\mathbf{n}+\mathbf{e}_2}, \sigma_{3,\mathbf{n}} \big| \mathbf{R}_\mathbf{n} \big| \sigma_{1,\mathbf{n}+\mathbf{e}_1}, \sigma_{2,\mathbf{n}}, \sigma_{3,\mathbf{n}+\mathbf{e}_3} \big\rangle. \quad (43)$$

Here the six external multi-spin variables (i.e. the indices of the matrix $\mathfrak{R}_{123}$) are associated with the six surfaces of the cube:

$$\vec{\sigma}_1 = \{\sigma_{1:0,n_2,n_3}\} \qquad \vec{\sigma}_2 = \{\sigma_{2:n_1,N_2,n_3}\} \qquad \vec{\sigma}_3 = \{\sigma_{3:n_1,n_2,0}\} \qquad n_j = 0, \ldots, N_j - 1 \quad (44)$$

and

$$\vec{\sigma}_1' = \{\sigma_{1:N_1,n_2,n_3}\} \qquad \vec{\sigma}_2' = \{\sigma_{2:n_1,0,n_3}\} \qquad \vec{\sigma}_3' = \{\sigma_{3:n_1,n_2,N_3}\} \qquad n_j = 0, \ldots, N_j - 1 \quad (45)$$

and the summation in (43) is taken with respect to all internal indices $\sigma_{j,\mathbf{n}}$. Anticipating what will be needed in (62) in order to prove that the fused weights satisfy a MTE of the same form as we had in (42), we use a reversed numbering for σ_2 and σ_2', so that the 'initial' external indices are $\sigma_1 = 0, \sigma_2 = N_2, \sigma_3 = 0$. This reversed numbering in the second space is also dictated by our choice of the line orientations in the lattice and, as a consequence of this, in the auxiliary plane (see figure 4).

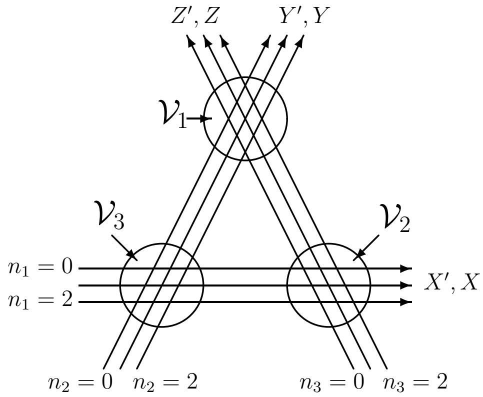

Figure 5. Ordering of the indices of the matrix $\mathfrak{R}_{123}$, shown for the case $N_1 = N_2 = N_3 = 3$ by drawing the auxiliary triangle in the auxiliary plane, as in the bottom part of figure 4.

In our next step we want to parametrize all $\mathbf{R_n}$ in (43) such that the fused weight $\mathfrak{R}$ again satisfies a modified tetrahedron equation. We shall show that using a theta-function parametrization this is possible and the $\mathfrak{R}_{ijk}$ obtained will depend on $6(N_1 + N_2 + N_3)$ free parameters.

We again use the generic algebraic curve Γ_g, and one vector $\mathbf{v} \in \mathbb{C}^g$. As in (30) each $\mathbf{R_n}$ will depend on three pairs of points on Γ_g, and to each $\mathbf{R_n}$ we assign different three pairs:

$$X'_{n_1}, X_{n_1}, Y'_{n_2}, Y_{n_2}, Z'_{n_3}, Z_{n_3} \qquad n_j = 0, \ldots, N_j - 1. \tag{46}$$

However, the argument $\mathbf{v}$ will be shifted for each $\mathbf{R_n}$ by an amount $\mathbf{I_n}$ which depends on the points assigned to 'previous' neighbours: We define

$$\mathbf{R}_{\mathbf{n}}^{(123)} = \mathbf{R}\big(\mathbf{v} + \mathbf{I_n}; X'_{n_1}, X_{n_1}; Y'_{n_2}, Y_{n_2}; Z'_{n_3}, Z_{n_3}\big) \qquad n_j = 0, \ldots, N_j - 1 \tag{47}$$

where

$$\mathbf{I_n} = \sum_{m_1=0}^{n_1-1} \mathbf{I}_{X_{m_1}}^{X'_{m_1}} + \sum_{m_2=0}^{n_2-1} \mathbf{I}_{Y_{m_2}}^{Y'_{m_2}} + \sum_{m_3=0}^{n_3-1} \mathbf{I}_{Z_{m_3}}^{Z'_{m_3}}. \tag{48}$$

Now (43) and (47) define the matrix function

$$\mathfrak{R}_{123}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; X', X; Y', Y; Z', Z) \tag{49}$$

where X', X, Y', Y, Z', Z stand for the ordered lists of divisors,

$$X = \big(X_0, X_1, \ldots, X_{N_1-1}\big) \qquad X' = \big(X'_0, X'_1, \ldots, X'_{N_1-1}\big) \qquad Y = (Y_0, Y_1, \ldots), \text{ etc.} \tag{50}$$

As to the index structure, recall (9),

$$\mathfrak{R}_{123} \in \mathrm{End}\,(\mathcal{V}^{N_2N_3} \otimes \mathcal{V}^{N_1N_3} \otimes \mathcal{V}^{N_1N_2}) \tag{51}$$

where in the same way as before we enumerate the number of $\mathcal{V}^{N_jN_k}$ in the tensor product (e.g., (43) are the matrix elements of $\mathfrak{R}_{123}$). In figure 5 we show the intersection lines of the planes of a $N_1 \times N_2 \times N_3$ cube which appear in an auxiliary plane (as in the bottom part of figure 4), which intersects the 'initial' edges corresponding to (44). On the section the $N_1 + N_2 + N_3$ planes become lines, and the edges of the cubic lattice become the intersection points of $N_1N_2 + N_2N_3 + N_1N_3$ lines in the auxiliary plane. The intersection points are gathered into three sets $\mathcal{V}_1 = \mathcal{V}^{N_2N_3}$ etc, and the index of $\mathcal{V}_j$ is the number of the corresponding $\mathcal{V}^{N_kN_l}$ in the tensor product in (51). Figure 5 helps arrange the numbering in (48) and the correct assignment of X'_{n_1}, X_{n_1}, etc.

3.2. The modified tetrahedron equation for the fused weights

For writing the MTE, apart from the three pairs of sets X', X; Y', Y; Z', Z of (49) and (50) we need a fourth pair

$$U = \big(U_0, U_1, \ldots, U_{N_0-1}\big) \qquad U' = \big(U'_0, U'_1, \ldots, U'_{N_0-1}\big).$$

Applying definition (43), in addition to (49) we construct the matrices

$$\mathfrak{R}_{145}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; U', U; Y', Y; Z', Z)$$
$$\mathfrak{R}_{246}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; U', U; X', X; Z', Z)$$
$$\mathfrak{R}_{356}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; U', U; X', X; Y', Y).$$

Their index structure is defined by

$$\begin{array}{lll} \mathcal{V}_1 = \mathcal{V}^{N_2N_3} & \mathcal{V}_2 = \mathcal{V}^{N_3N_1} & \mathcal{V}_3 = \mathcal{V}^{N_1N_2} \\ \mathcal{V}_4 = \mathcal{V}^{N_0N_3} & \mathcal{V}_5 = \mathcal{V}^{N_0N_2} & \mathcal{V}_6 = \mathcal{V}^{N_0N_1} \end{array} \tag{52}$$

so that, e.g., $\mathfrak{R}_{145}$ is acting in a space of dimension $N^{N_2N_3+N_0N_3+N_0N_2}$. For $\mathfrak{R}_{145}$ in analogy to definitions (47) and (48) one uses

$$\mathbf{R}^{(145)}_{\mathbf{n}} = \mathbf{R}\big(\mathbf{v} + \mathbf{I}_{\mathbf{n}}; U'_{n_0}, U_{n_0}; Y'_{n_2}, Y_{n_2}; Z'_{n_3}, Z_{n_3}\big) \tag{53}$$

with

$$\mathbf{I}_{\mathbf{n}} = \sum_{m_0=0}^{n_0-1} \mathbf{I}^{U'_{m_0}}_{U_{m_0}} + \sum_{m_2=0}^{n_2-1} \mathbf{I}^{Y'_{m_2}}_{Y_{m_2}} + \sum_{m_3=0}^{n_3-1} \mathbf{I}^{Z'_{m_3}}_{Z_{m_3}} \tag{54}$$

similarly for $\mathfrak{R}_{246}$ and $\mathfrak{R}_{356}$.

Theorem 2. *The matrices* $\mathfrak{R}$ *defined in (43) obey the modified tetrahedron equation*

$$\begin{aligned} &\mathfrak{R}_{123}(\mathbf{v})\mathfrak{R}_{145}(\mathbf{v}+\mathbf{I}_X)\mathfrak{R}_{246}(\mathbf{v})\mathfrak{R}_{356}(\mathbf{v}+\mathbf{I}_Z) \\ &\qquad = \rho\mathfrak{R}_{356}(\mathbf{v})\mathfrak{R}_{246}(\mathbf{v}+\mathbf{I}_Y)\mathfrak{R}_{145}(\mathbf{v})\mathfrak{R}_{123}(\mathbf{v}+\mathbf{I}_U) \end{aligned} \tag{55}$$

where

$$\begin{aligned} \mathbf{I}_U &= \sum_{n_0=0}^{N_0-1} \mathbf{I}^{U'_{n_0}}_{U_{n_0}} \qquad \mathbf{I}_X = \sum_{n_1=0}^{N_1-1} \mathbf{I}^{X'_{n_1}}_{X_{n_1}} \\ \mathbf{I}_Y &= \sum_{n_2=0}^{N_2-1} \mathbf{I}^{Y'_{n_2}}_{Y_{n_2}} \qquad \mathbf{I}_Z = \sum_{n_3=0}^{N_3-1} \mathbf{I}^{Z'_{n_3}}_{Z_{n_3}}. \end{aligned} \tag{56}$$

Proof. The main content of this theorem is the appearance of the specific set of shifts (48) and (56). For the proof it is convenient to introduce some compact notations. Instead of using the number labels for the $\mathfrak{R}$ we shall use the labels U, X, Y, Z just as these were introduced in (36) and (42) for the single vertex matrices $\mathbf{R}$. So, for the box $\mathfrak{R}$-matrix and its sets of divisors we write,

$$\mathfrak{R}_{123}(\mathbf{v}) \Longrightarrow \mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}) \qquad \mathfrak{R}_{145}(\mathbf{v}) \Longrightarrow \mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}) \qquad \text{etc.} \tag{57}$$

In this short notation, formulae (43) and (47) imply the definition

$$\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}) = \prod_{n_1=0\uparrow N_1-1} \prod_{n_2=N_2-1\downarrow 0} \prod_{n_3=0\uparrow N_3-1} \mathbf{R}^{X_{n_1}Y_{n_2}Z_{n_3}}(\mathbf{v}+\mathbf{I}_{\mathbf{n}}) \tag{58}$$

where we use ordered products

$$\prod_{n_1=0\uparrow N_1-1} \mathfrak{f}_{n_1} = \mathfrak{f}_0\mathfrak{f}_1\cdots\mathfrak{f}_{N_1-1} \qquad \prod_{n_2=N_2-1\downarrow 0} \mathfrak{f}_{n_2} = \mathfrak{f}_{N_2-1}\cdots\mathfrak{f}_1\mathfrak{f}_0. \tag{59}$$

For the triple (X, Y, Z), $\mathbf{I_n}$ is given by (48). Each $\mathbf{R}^{\mathsf{X}_{n_1}\mathsf{Y}_{n_2}\mathsf{Z}_{n_3}}$ acts non-trivially in only three of all the spaces (52) and the ordering is relevant just for neighbouring indices X_{n_1} or Y_{n_2} or Z_{n_3}. The analogous modifications required to get the other matrices $\mathfrak{R}_{145}$, etc should be evident.

Now we turn to the proof of the theorem which will be by mathematical induction. The theorem claims the validity of the MTE for arbitrary N_0, N_1, N_2, N_3. For the initial point $N_0 = N_1 = N_2 = N_3 = 1$ the MTE holds because it is just (42). Then to prove the theorem, we reduce the MTE (55) for some N_j to MTEs with $N_j' \leqslant N_j$. Thus one has four similar steps of the induction. Here we consider the induction step for the X-direction; the other steps follow analogously.

We split the list X into two sublists of length $N_1^{(1)}$ and $N_1^{(2)} = N_1 - N_1^{(1)}$:

$$X^{(1)} = \big(X_0, X_1, \ldots, X_{N_1^{(1)}-1}\big) \qquad X^{(2)} = \big(X_{N_1^{(1)}}, \ldots, X_{N_1-1}\big) \tag{60}$$

so that $\mathbf{I}_{X^{(1)}} = \sum_{n_1=0}^{N_1^{(1)}-1} \mathbf{I}_{X_{n_1}}$ and $\mathbf{I}_{X^{(2)}} = \mathbf{I}_X - \mathbf{I}_{X^{(1)}}$. According to this splitting and due to definition (58)

$$\begin{aligned}
&\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}) = \mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\\
&\mathfrak{R}^{\mathsf{UXZ}}(\mathbf{v}) = \mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v})\\
&\mathfrak{R}^{\mathsf{UXY}}(\mathbf{v}) = \mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v}).
\end{aligned} \tag{61}$$

The meaning of the notation $\mathsf{X}^{(1)}$ and $\mathsf{X}^{(2)}$ used in (61) should be evident from (58). Observe that because of the reverse numbering with respect to the middle superscript of $\mathfrak{R}$ in (58), the factors in the latter two equations appear in reverse order. Since $\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})$ contains no X, neither as subscript nor in the argument, it will not be split. However, we have to put

$$\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_X) = \mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_{X^{(2)}}).$$

Now substituting (61) into the LHS of (55) written in our new superscript notation, and abbreviating $\mathbf{v}_1 \equiv \mathbf{v} + \mathrm{I}_{X^{(1)}}$, we get

$$\begin{aligned}
&\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_X)\mathfrak{R}^{\mathsf{UXZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{UXY}}(\mathbf{v}+\mathbf{I}_Z)\\
&\quad = \mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_{X^{(2)}})\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\\
&\qquad \times \mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_Z)\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_Z)\\
&\quad = \mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v})\big[\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}_1)\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}_1+\mathbf{I}_{X^{(2)}})\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}_1)\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}_1+\mathbf{I}_Z)\big]\\
&\qquad \times \mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_Z)\\
&\quad = \mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v})\big[\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}_1)\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}_1+\mathbf{I}_Y)\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}_1)\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}_1+\mathbf{I}_U)\big]\\
&\qquad \times \mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_Z)\\
&\quad = \mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}_1)\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}_1+\mathbf{I}_Y)\\
&\qquad \times \big[\mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+X^{(1)})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_Z)\big]\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}_1+\mathbf{I}_U)\\
&\quad = \mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}_1)\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}_1+\mathbf{I}_Y)\\
&\qquad \times \big[\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v}+\mathbf{I}_Y)\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v}+\mathbf{I}_U)\big]\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}_1+\mathbf{I}_U)\\
&\quad = \mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Y}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Y}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX}^{(2)}\mathsf{Z}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_Y)\mathfrak{R}^{\mathsf{UX}^{(1)}\mathsf{Z}}(\mathbf{v}+\mathbf{I}_Y)\\
&\qquad \times \mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{X}^{(1)}\mathsf{YZ}}(\mathbf{v}+\mathbf{I}_U)\mathfrak{R}^{\mathsf{X}^{(2)}\mathsf{YZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_U)\\
&\quad = \mathfrak{R}^{\mathsf{UXY}}(\mathbf{v})\mathfrak{R}^{\mathsf{UXZ}}(\mathbf{v}+\mathbf{I}_Y)\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}+\mathbf{I}_U).
\end{aligned} \tag{62}$$

From the third to the fourth line of (62) within the inserted brackets we used the MTE for the smaller set $(U, X^{(2)}, Y, Z)$. In order to isolate the terms containing $X^{(2)}$ the order of factors in the second line of (61), which was used in the first step, is crucial. Going from the fifth to the sixth line in the brackets we used the MTE for $(U, X^{(1)}, Y, Z)$. In the other steps of (62) we just commuted or combined various terms. The last step is made possible by the 'reverse' order of factors in the last line of (61). □

4. Special cases: solving the tetrahedron equation

4.1. Compact algebraic curve

Now let $N_0 = N_1 = N_2 = N_3 = M$. Starting from the generic parametrization of the MTE (55), the usual tetrahedron equation is obtained if

$$\mathfrak{R}(\mathbf{v}) \equiv \mathfrak{R}(\mathbf{v}+\mathbf{I}) \qquad \text{and} \qquad \rho = 1. \tag{63}$$

This is the case when

$$\mathbf{I}_U = \mathbf{I}_X = \mathbf{I}_Y = \mathbf{I}_Z = 0 \qquad \text{on} \qquad \mathrm{Jac}(\Gamma_g) \tag{64}$$

and the ratios of θ-functions (25) are periodical.

According to Abel's theorem, (64) means that there are four meromorphic functions u, x, y, z on Γ_g with the divisors

$$\begin{aligned} (u) &= \sum_{n_0=0}^{M-1} U'_{n_0} - U_{n_0} \qquad & (x) &= \sum_{n_1=0}^{M-1} X'_{n_1} - X_{n_1} \\ (y) &= \sum_{n_2=0}^{M-1} Y'_{n_2} - Y_{n_2} \qquad & (z) &= \sum_{n_3=0}^{M-1} Z'_{n_3} - Z_{n_3}. \end{aligned} \tag{65}$$

As is well known (see, e.g., theorem 10-23 of [20]), conditions (64) are a strong restriction for the type of Γ_g: Γ_g is the algebraic curve given by the polynomial equation

$$P(x, y) \stackrel{\text{def}}{=} \sum_{a,b=0}^{M} x^a y^b p_{a,b} = 0. \tag{66}$$

The choice of any pair of x, y, z, u produces an equivalent polynomial equation. The form of the polynomial $P(x, y)$ provides the restriction for the genus,

$$g \leqslant (M-1)^2. \tag{67}$$

Thus we come to

Theorem 3. *Let Γ_g be the compact algebraic curve defined by the polynomial equation (66). Let four sets of $U'_{n_0}, U_{n_0}, X'_{n_1}, X_{n_1}, Y'_{n_2}, Y_{n_2}$ and Z'_{n_3}, Z_{n_3}, $n_k = 0, \ldots, M-1$, be the divisors of four meromorphic functions u, x, y, z on Γ_g. Then the tetrahedron equation is satisfied*

$$\begin{aligned} &\mathfrak{R}_{123}(x, y, z)\mathfrak{R}_{145}(u, y, z)\mathfrak{R}_{246}(u, x, z)\mathfrak{R}_{356}(u, x, y) \\ &\qquad = \mathfrak{R}_{356}(u, x, y)\mathfrak{R}_{246}(u, x, z)\mathfrak{R}_{145}(u, y, z)\mathfrak{R}_{123}(x, y, z) \end{aligned} \tag{68}$$

where four matrices are the same matrix function of different arguments,

$$\mathfrak{R}(x, y, z) = \mathfrak{R}(\mathbf{v}; X', X; Y', Y; Z', Z) \qquad \text{etc} \tag{69}$$

defined via (43), (47), (49) and (65).

According to the conventional terminology, one may say that u, x, y, z are the spectral parameters, the moduli of Γ_g are the moduli of the tetrahedron equation and vector $\mathbf{v}$ is a kind of deformation parameter.

Theorem 3 may also be proved differently, without mentioning the simple MTE. In this alternative approach one considers the auxiliary linear problem for the whole box and the corresponding mappings. See, e.g., [21, 22] for the description of this method and [23] for the parametrization of the classical equations of motion. Remarkably, in this alternative way the spectral curve (66) appears naturally from the linear problem.

4.2. Simple tetrahedron equation for the ZBB case recovered

In section 1 we discussed the ZBB model and its $\mathbf{R}$-matrix (7). We now show how our scheme contains this case. Formally ZBB's tetrahedron equation corresponds to (68) with $M = 1$. It gives $g = 0$, i.e. the spectral curve is the sphere with

$$E(X, Y) = \frac{X - Y}{\sqrt{\mathrm{d}X\,\mathrm{d}Y}} \qquad \Theta(\,) \equiv 1. \tag{70}$$

Here the formal theta function has no argument since the Jacobian is zero-dimensional. Conditions (64) therefore are out of use, and the parametrization (30) and (31) contains the Nth roots of the cross-ratios such as $\frac{(X-Z)(X'-Z')}{(X'-Z)(X-Z')}$. One may show that the number of independent cross-ratios is the number of variables $X, X', \ldots$ *minus* three. Therefore, the single $\mathbf{R}$-matrix contains $6 - 3 = 3$ independent complex parameters (as it should), and the simple tetrahedron equation contains $8 - 3 = 5$ independent complex parameters (again as it should). It means that the tetrahedral condition, which appears in Zamolodchikov's parametrization of $\mathbf{R}$ in terms of spherical triangles, is taken into account automatically. Moreover, the parametrization with the help of the cross-ratios automatically takes into account the geometric structure of any set of planes in three-dimensional Euclidean space. The parametrization of the inhomogeneous Zamolodchikov–Bazhanov–Baxter model in terms of cross-ratios corresponding to the $g = 0$ limit of (30), (31) and (47) has already been used in [24].

4.3. Chessboard model

Previously derived 'chessboard models' of the lattice statistical mechanics based on the modified tetrahedron equation [10, 11] are of course related to our considerations. The term 'chessboard' appeared as the visual interpretation of the cubic lattice with $M = 2$ being homogeneous. It means that the cubic lattice consists of eight different types of vertices (i.e. eight different types of Boltzmann weights)—a kind of three-dimensional analogue of the chessboard with eight different colours of the cells.

The models described in [10, 11] are at first the so-called IRC-type models, but with the help of vertex–IRC correspondence [8] one may construct their vertex reformulation. Thus the model implicitly described in [11] is equivalent to $M = 2$, $g = 1$ of our scheme. For $g = 1$ the curve and its Jacobian are isomorphic, so that without loss of generality one may chose

$$\mathbf{I}_Y^X = \mathtt{X} - \mathtt{Y} \in \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau. \tag{71}$$

Further, one may use θ_1

$$\Theta(v) = \theta_1(v, \tau) \equiv \sum_{n=-\infty}^{\infty} \exp\left[\mathrm{i}\pi\tau(n + 1/2)^2 + 2\mathrm{i}\pi(v + 1/2)(n + 1/2)\right] \tag{72}$$

as the basic theta function, and $E(X,Y) \sim \theta_1(\mathtt{X}-\mathtt{Y})$ as the prime form. These formulae simplify definitions (25). Periodicity conditions (65) may be chosen

$$\mathtt{X}_0' - \mathtt{X}_0 + \mathtt{X}_1' - \mathtt{X}_1 = \mathtt{Y}_0' - \mathtt{Y}_0 + \mathtt{Y}_1' - \mathtt{Y}_1 = \mathtt{Z}_0' - \mathtt{Z}_0 + \mathtt{Z}_1' - \mathtt{Z}_1 = \mathtt{U}_0' - \mathtt{U}_0 + \mathtt{U}_1' - \mathtt{U}_1 = 1. \quad (73)$$

Note, the 1 in the right-hand side of (73) is equivalent to τ (due to the Jacobi transform), while 0 instead of 1 in the right-hand side of (73) gives a trivial model.

The model explicitly described before in [10] corresponds to $M = 2$, $g = 1$, $\mathtt{X}_0 = \mathtt{X}_1$, $\mathtt{X}_0' = \mathtt{X}_1'$, $\mathtt{Y}_0 = \mathtt{Y}_1$, $\mathtt{Y}_0' = \mathtt{Y}_1'$, $\mathtt{Z}_0 = \mathtt{Z}_1$, $\mathtt{Z}_0' = \mathtt{Z}_1{}'$ etc with the condition (73). This choice leads to the identification of the parameters in (43)

$$\mathbf{R_n} = \mathbf{R_{n+e_1+e_2}} = \mathbf{R_{n+e_1+e_3}} = \mathbf{R_{n+e_2+e_3}} \quad (74)$$

so the cells of this three-dimensional chessboard have only two ‘colours’.

Note that the vertex–IRC duality is not exact because it changes the boundary conditions.

5. Conclusions

We considered a large class of integrable 3D lattice models which have Weyl variables at *N*th root of unity as dynamic variables. We have shown how the Boltzmann weights can be conveniently parametrized in terms of *N*th roots of theta functions on a Jacobian of a compact Riemann surface. The Fermat relations of the points determining the Boltzmann weights are simple Fay identities and the classical equations determining the scalar parameters are a consequence of a double-Fay identity. In the modified tetrahedron equation we have four pairs of arbitrary points on the Riemann surface in simple permuted combinations.

This parametrization allows a compact formulation of the rules to form fused Boltzmann weights $\mathfrak{R} \in \mathrm{End}\, \mathbb{C}^{3NM^2}$ which are the partition functions of open boxes of arbitrary size. The $\mathfrak{R}$ obey the modified tetrahedron equation and are again parametrized terms on *N*th roots of theta functions on the Jacobian of a genus $g = (M-1)^2$ compact Riemann surface Γ_g. The spectral parameters of the vertex weight $\mathfrak{R}$ are three meromorphic functions on the spectral curve Γ_g. For the case that the Jacobi transforms become trivial the $\mathfrak{R}$ obey the simple tetrahedron equation. The Zamolodchikov–Baxter–Bazhanov model and the chessboard model are obtained as special cases.

So, the scheme discussed here contains and generalizes many known 3D integrable models. The hope is that the framework is now sufficiently general to contain physically interesting models with a non-trivial phase structure. However, to get information on partition functions, either analytically or approximately, is still a very difficult open problem. There is no way known to generalize Baxter’s special method [26] by which he obtained the partition function of the ZBB model.

Acknowledgments

SP is grateful to Bonn and Angers Universities, where the work was done partially. SS thanks the Max-Planck-Institut für Mathematik and Bonn University. This work has been supported in part by the contract INTAS OPEN 00-00055 and by the Heisenberg–Landau programme HLP-2002-11. SP’s work was supported in part by the grants CRDF RM1-2334-MO-02, RFBR 03-02-17373 and the grant for support of scientific schools 1999.2003.2. SS’s work was supported in part by the grants CRDF RM1-2334-MO-02 and RFBR 01-01-00201.

References

[1] Zamolodchikov A B 1981 *Commun. Math. Phys.* **79** 489
[2] Baxter R J 1983 *Commun. Math. Phys.* **88** 185

[3] Bazhanov V V and Baxter R J 1992 *J. Stat. Phys.* **69** 453
Bazhanov V V and Baxter R J 1993 *J. Stat. Phys.* **71** 453
[4] Mangazeev V, Kashaev R and Stroganov Yu 1993 *Int. J. Mod. Phys.* A **8** 587, 1399
[5] Sergeev S M, Mangazeev V V and Stroganov Yu G 1996 *J. Stat. Phys.* **82** 31
[6] Korepanov I G 1993 *Commun. Math. Phys.* **154** 85
[7] Hietarinta J 1994 *J. Phys. A: Math. Gen.* **27** 5727
[8] Boos H E, Mangazeev V V, Sergeev S M and Stroganov Yu G 1996 *Mod. Phys. Lett.* A **11** 491
[9] von Gehlen G, Pakuliak S and Sergeev S 2003 *J. Phys. A: Math. Gen.* **36** 975
[10] Mangazeev V V and Stroganov Yu G 1993 *Mod. Phys. Lett.* A **8** 3475
Mangazeev V V, Sergeev S M and Stroganov Yu G 1994 *Int. J. Mod. Phys.* A **9** 5517
[11] Boos H E, Mangazeev V V and Sergeev S M 1995 *Int. J. Mod. Phys.* A **10** 4041
[12] Maillard J-M and Sergeev S M 1997 *Phys. Lett.* B **405** 55
[13] Mumford D 1983 1985 *Tata Lectures of Theta I, II* (Boston, MA: Birkhauser)
[14] Krichever I 1978 *Russ. Math. Survey* **33** 215
[15] Shitota T 1986 *Inv. Math.* **83** 333
[16] Mulase M 1984 *J. Differ. Geom.* **19** 403
[17] Krichever I, Wiegmann P and Zabrodin A 1998 *Commun. Math. Phys.* **193** 373
[18] Sergeev S M 2000 *J. Nonlinear Math. Phys.* **7** 57
[19] Baxter R J and Forrester P J 1985 *J. Phys. A: Math. Gen.* **18** 1483
[20] Springer G 1981 *Introduction to Riemann Surfaces* (New York: Chelsea)
[21] Sergeev S M 1999 *J. Phys. A: Math. Gen.* **32** 5639
[22] Sergeev S M 2001 Integrable three dimensional models in wholly discrete space-time *Integrable Structures of Exactly Solvable Two-Dimensional Models etc (NATO Science Series II)* ed S Pakuliak and G von Gehlen vol 35 (Dordrecht: Kluwer) pp 293–304
[23] Pakuliak S and Sergeev S 2002 Spectral curves and parametrizations of a discrete integrable 3-dimensional model *Preprint* MPI 02-85, nlin.SI/0209019
[24] Sergeev S 2002 Functional equations and quantum separation of variables for 3D spin models *Preprint* MPI 02-46
[25] von Gehlen G, Pakuliak S and Sergeev S 2003 The modified tetrahedron equation and its solutions *Preprint* nlin.SI/0303043
[26] Baxter R J 1986 *Physica* D **18** 321